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Quasi M-convex and L-convex functions—quasiconvexity in discrete optimization☆

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Abstract

We introduce two classes of discrete quasiconvex functions, called quasi M- and L-convex functions, by generalizing the concepts of M- and L-convexity due to Murota (Adv. Math. 124 (1996) 272) and (Math. Programming 83 (1998) 313). We investigate the structure of quasi M- and L-convex functions with respect to level sets, and show that various greedy algorithms work for the minimization of quasi M- and L-convex functions.

Keywords: Quasiconvex function; Discrete optimization; Matroid; Base polyhedron

1. Introduction

The concept of convexity for sets and functions plays a central role in continuous optimization (or nonlinear programming with continuous variable), and has various applications in the areas of mathematical economics, engineering, operations research, etc. [2,17,20]. Many important applications of convexity in optimization rely on the fact that a local minimum of a convex function is also a global minimum. Due to this property, we can find a global minimum of a convex function by iteratively moving in descent directions, i.e., so-called descent algorithms work for the convex function minimization. Therefore, convexity for a function is a sufficient condition for the success

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of descent methods. Most descent methods, however, work for a fairly larger class of functions called quasiconvex functions.

A function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) \leqslant \max\{f(x), f(y)\} \quad (\forall x, y \in \mathrm{dom}\, f,\ \forall \alpha \in (0,1)),$$

and semistrictly quasiconvex if it satisfies

$$f(\alpha x + (1-\alpha)y) < \max\{f(x), f(y)\}$$

$$(\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) \neq f(y),\ \forall \alpha \in (0,1)),$$

where $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}$. It is easy to see that convexity implies semistrict quasiconvexity, and semistrict quasiconvexity implies quasiconvexity under a certain assumption. Although (semistrict) quasiconvexity is a weaker property than convexity, it still has nice properties as follows:

- A strict local minimum of a quasiconvex function is also a strict global minimum.
- A local minimum of a semistrictly quasiconvex function is also a global minimum.
- Level sets of quasiconvex functions are convex sets.

Due to these properties, quasiconvexity also plays an important role in continuous optimization. See [1] for more accounts on quasiconvexity.

In the area of discrete optimization, on the other hand, discrete analogues of convexity, or "discrete convexity" for short, have been considered, with a view to identifying the discrete structure that guarantees the success of descent methods, i.e., the so-called "greedy algorithms". Examples of discrete convexity are "discretely convex functions" by Miller [8], "integrally convex functions" by Favati–Tardella [3], and "M-convex and L-convex functions" by Murota [10–14] as well as their variants called "M$^\natural$-convex functions" by Murota–Shioura [15] and "L$^\natural$-convex functions" by Fujishige–Murota [4].

Let V be a finite set. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called M-convex if $\mathrm{dom}\, f = \{x \in \mathbf{Z}^V \mid f(x) < +\infty\}$ is nonempty and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f,\ \forall u \in \mathrm{supp}^+(x-y),\ \exists v \in \mathrm{supp}^-(x-y)$:

$$f(x) + f(y) \geqslant f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v), \tag{1.1}$$

where $\mathrm{supp}^+(x-y) = \{w \in V \mid x(w) > y(w)\}$, $\mathrm{supp}^-(x-y) = \{w \in V \mid x(w) < y(w)\}$, and $\chi_w \in \{0,1\}^V$ is the characteristic vector of $w \in V$. A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (SBM) and (TRF):

(SBM) g is submodular, i.e., $g(p) + g(q) \geqslant g(p \wedge q) + g(p \vee q)$ for all $p, q \in \mathbf{Z}^V$,

(TRF) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r$ $(\forall p \in \mathbf{Z}^V,\ \forall \lambda \in \mathbf{Z})$,

where $p \wedge q, p \vee q \in \mathbf{Z}^V$ are defined by $(p \wedge q)(w) = \min\{p(w), q(w)\}$, $(p \vee q)(w) = \max\{p(w), q(w)\}$ $(w \in V)$.

M- and L-convex functions have various nice properties as discrete convex functions:

(i) A local minimum of an M-/L-convex function is also a global minimum.
(ii) M-/L-convex functions can be extended to ordinary convex functions.

Table 1
Possible sign patterns of $\Delta f(x;v,u)$ and $\Delta f(y;u,v)$ in (M-EXC)

$\Delta f(x;v,u) \setminus \Delta f(y;u,v)$	$-$	0	$+$
$-$	○	○	○
0	○	○	×
$+$	○	×	×

○ ··· possible, × ··· impossible

(iii) Various duality theorems hold.
(iv) M- and L-convex functions are conjugate to each other.

In particular, property (i) shows that greedy algorithms work for the M-/L-convex function minimization. However, we see from results in continuous optimization that strong properties such as M-/L-convexity are not required for the success of greedy algorithms, and that some properties like "quasi M-/L-convexity" will suffice.

The main aim of this paper is to introduce the concepts of quasi M- and L-convex functions by generalizing those of M- and L-convexity.

To extend the concept of M-convexity to quasi M-convexity, we relax condition (1.1) while keeping the possible sign patterns of values $\Delta f(x;v,u) = f(x - \chi_u + \chi_v) - f(x)$ and $\Delta f(y;u,v) = f(y + \chi_u - \chi_v) - f(y)$ in mind. Table 1 shows the possible sign patterns of those values for an M-convex function.

We call f *quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (QM):

(QM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$\Delta f(x;v,u) \leqslant 0 \quad \text{or} \quad \Delta f(y;u,v) \leqslant 0.$$

Similarly, we call f *semistrictly quasi M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and it satisfies (SSQM):

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

(i) $\Delta f(x;v,u) \geqslant 0 \Rightarrow \Delta f(y;u,v) \leqslant 0$, and
(ii) $\Delta f(y;u,v) \geqslant 0 \Rightarrow \Delta f(x;v,u) \leqslant 0$.

We introduce the concept of quasi L-convex functions by generalizing the submodularity of functions to quasisubmodularity. We consider two different generalizations of the submodularity:

(QSB) For all $p, q \in \mathbf{Z}^V$ we have $g(p \wedge q) \leqslant g(p)$ or $g(p \vee q) \leqslant g(q)$.

(SSQSB) For all $p, q \in \mathbf{Z}^V$ we have both

(i) $g(p \vee q) \geqslant g(q) \Rightarrow g(p \wedge q) \leqslant g(p)$, and
(ii) $g(p \wedge q) \geqslant g(p) \Rightarrow g(p \vee q) \leqslant g(q)$.

We call a function g quasisubmodular (resp. semistrictly quasisubmodular) if it satisfies (QSB) (resp. (SSQSB)). Similarly, we call g quasi L-convex (resp. semistrictly quasi L-convex) if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (QSB) (resp. (SSQSB)) and (TRF).

The organization of this paper is as follows. In Section 2, we explain some definitions and notation used in this paper. In Sections 3 and 5, we show some properties of level sets of quasi M-/L-convex functions and prove that the classes of quasi M-/L-convex functions are closed under various fundamental operations. These results justify the definitions of quasi M-/L-convexity. Finally, we show that various greedy algorithms work for the minimization of (semistrictly) quasi M-/L-convex functions in Sections 4 and 6. We also establish some proximity theorems for (semistrictly) quasi M-/L-convex functions, which guarantee the applicability of the so-called "scaling technique" to the quasi M-/L-convex function minimization.

The concepts of M$^\natural$-convexity by Murota–Shioura [15] and L$^\natural$-convexity by Fujishige–Murota [4] can be also extended to quasi M$^\natural$-/L$^\natural$-convexity, and the results in this paper can be restated in obvious ways in terms of quasi M$^\natural$-/L$^\natural$-convex functions.

Remark 1.1. Condition (SSQSB) was introduced by Milgrom–Shannon [7], in which $g:\mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$ is called quasisupermodular if $-g$ satisfies (SSQSB) above. We adopt the terminology "semistrict quasisubmodularity" for the property (SSQSB) in view of our results shown in Section 5.

Remark 1.2. In [22], Zimmermann considers combinatorial optimization problems with quasiconvex objective functions in real variables.

2. Preliminaries

We denote by $\mathbf{R}$ the set of reals, by $\mathbf{Z}$ the set of integers, and by $\mathbf{R}_{++}$ the set of positive reals. For any finite set X, its cardinality is denoted by $|X|$. Throughout this paper, we assume that V is a nonempty finite set with $|V|=n(>0)$. The characteristic vector of a subset $X\subseteq V$ is denoted by χ_X ($\in\{0,1\}^V$), i.e., $\chi_X(w)=1$ for $w\in X$ and $\chi_X(w)=0$ for $w\in V\setminus X$. In particular, we use the notation $\mathbf{0}=\chi_\emptyset$, $\mathbf{1}=\chi_V$. For $x=(x(w)\mid w\in V)\in\mathbf{R}^V$, we define $\|x\|_1=\sum_{v\in V}|x(v)|$, $\langle p,x\rangle=\sum_{v\in V}p(v)x(v)$ ($p\in\mathbf{R}^V$), and $x(X)=\sum_{v\in X}x(v)$ ($X\subseteq V$).

For $a:V\to\mathbf{Z}\cup\{-\infty\}$ and $b:V\to\mathbf{Z}\cup\{+\infty\}$ with $a(v)\leqslant b(v)$ ($v\in V$), we define the interval $[a,b]$ by $[a,b]=\{x\in\mathbf{Z}^V\mid a(v)\leqslant x(v)\leqslant b(v)\ (v\in V)\}$.

Let $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$. The set $\operatorname{dom} f=\{x\in\mathbf{Z}^V\mid f(x)<+\infty\}$ is called the *effective domain* of f. We denote by $\arg\min f$ the set of the minimizers of f, i.e., $\arg\min f=\{x\in\mathbf{Z}^V\mid f(x)\leqslant f(y)\ (\forall y\in\mathbf{Z}^V)\}$. For any $\alpha\in\mathbf{R}\cup\{+\infty\}$, the *level set* $\mathrm{L}(f,\alpha)$ is defined by $\mathrm{L}(f,\alpha)=\{x\in\mathbf{Z}^V\mid f(x)\leqslant\alpha\}$. For a set $S\subseteq\mathbf{Z}^V$, the *indicator function* $\delta_S:\mathbf{Z}^V\to\{0,+\infty\}$ of S is given by $\delta_S(x)=0$ ($x\in S$) and $\delta_S(x)=+\infty$ ($x\notin S$).

We define (semistrict) quasiconvexity for functions $\varphi:\mathbf{Z}\to\mathbf{R}\cup\{+\infty\}$ in the following way: we call a function φ *quasiconvex* if it satisfies

$$\varphi(\beta)\leqslant\max\{\varphi(\alpha_1),\varphi(\alpha_2)\}\quad(\forall\alpha_1,\alpha_2,\beta\in\mathbf{Z}\text{ with }\alpha_1<\beta<\alpha_2),\tag{2.1}$$

and *semistrictly quasiconvex* if it is a quasiconvex function and satisfies

$$\varphi(\beta)<\max\{\varphi(\alpha_1),\varphi(\alpha_2)\}\quad(\forall\alpha_1,\alpha_2,\beta\in\mathbf{Z}\text{ with }\alpha_1<\beta<\alpha_2,\ \varphi(\alpha_1)\neq\varphi(\alpha_2)).\tag{2.2}$$

Remark 2.1. For $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, semistrict quasiconvexity implies quasiconvexity under a certain assumption [1,2]. For $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$, on the other hand, property (2.2) alone does not imply the quasiconvexity in general. It is convenient for our subsequent development to assume quasiconvexity in the definition of semistrict quasiconvexity for φ.

Theorem 2.2. *Let* $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$.

(i) φ *is quasiconvex* $\Leftrightarrow$ *for any* $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ *with* $\alpha_1 < \alpha_2$ *we have* $\min\{\varphi(\alpha_1 + 1), \varphi(\alpha_2 - 1)\} \leqslant \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}$.

(ii) *Under quasiconvexity* (2.1), φ *satisfies* (2.2) $\Leftrightarrow$ *for any* $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ *with* $\alpha_1 < \alpha_2$ *and* $\varphi(\alpha_1) \neq \varphi(\alpha_2)$ *we have* $\min\{\varphi(\alpha_1 + 1), \varphi(\alpha_2 - 1)\} < \max\{\varphi(\alpha_1), \varphi(\alpha_2)\}$.

(iii) φ *is semistrictly quasiconvex* $\Leftrightarrow$ *for any* $\alpha_1, \alpha_2 \in \mathrm{dom}\,\varphi$ *with* $\alpha_1 < \alpha_2$ *we have both* $\varphi(\alpha_1 + 1) \geqslant \varphi(\alpha_1) \Rightarrow \varphi(\alpha_2 - 1) \leqslant \varphi(\alpha_2)$ *and* $\varphi(\alpha_2 - 1) \geqslant \varphi(\alpha_2) \Rightarrow \varphi(\alpha_1 + 1) \leqslant \varphi(\alpha_1)$.

3. Quasi M-convex functions

We first review the concept of M-convexity for sets and functions. A set $B \subseteq \mathbf{Z}^V$ is called *M-convex* if B is nonempty and satisfies

(B-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$:

$$x - \chi_u + \chi_v \in B \quad \text{and} \quad y + \chi_u - \chi_v \in B.$$

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *M-convex* if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC) (see Introduction for the property (M-EXC)).

M-convex sets and functions can be characterized by the (seemingly) weaker properties:

(B-EXC$_\mathrm{w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$:

$$x - \chi_u + \chi_v \in B \quad \text{and} \quad y + \chi_u - \chi_v \in B.$$

(M-EXC$_\mathrm{w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$ satisfying (1.1).

Theorem 3.1 (Tomizawa [21], Murota [10, Theorem 3.1]). *Let* $B \subseteq \mathbf{Z}^V$ *and* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then*, (i) (B-EXC) *for* $B \Leftrightarrow$ (B-EXC$_\mathrm{w}$) *for* B, *and* (ii) (M-EXC) *for* $f \Leftrightarrow$ (M-EXC$_\mathrm{w}$) *for* f.

3.1. Definitions of quasi M-convex functions

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *quasi M-convex* (resp. *semistrictly quasi M-convex*) if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (QM) (resp. (SSQM)):

(QM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$:

$$\Delta f(x; v, u) \leqslant 0 \quad \text{or} \quad \Delta f(y; u, v) \leqslant 0. \tag{3.1}$$

(SSQM) $\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$\text{(i)}\ \ \Delta f(x; v, u) \geqslant 0 \ \Rightarrow\ \Delta f(y; u, v) \leqslant 0, \quad \text{and}$$

$$\text{(ii)}\ \ \Delta f(y; u, v) \geqslant 0 \ \Rightarrow\ \Delta f(x; v, u) \leqslant 0. \tag{3.2}$$

Note that (SSQM) can be rewritten as follows:

$\forall x, y \in \mathrm{dom}\, f$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying at least one of (i) $\Delta f(x; v, u) < 0$, (ii) $\Delta f(y; u, v) < 0$, and (iii) $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$.

We also consider weaker properties than (QM) and (SSQM):

($\mathrm{QM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (3.1).

($\mathrm{SSQM_w}$) $\forall x, y \in \mathrm{dom}\, f$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (3.2).

The set version of quasi M-convexity can be obtained by translating the properties (QM) and ($\mathrm{QM_w}$) for the indicator function $\delta_B : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $B \subseteq \mathbf{Z}^V$ in terms of B.

(Q-EXC) $\forall x, y \in B$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$x - \chi_u + \chi_v \in B \quad \text{or} \quad y + \chi_u - \chi_v \in B.$$

($\mathrm{Q\text{-}EXC_w}$) $\forall x, y \in B$ with $x \neq y$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$:

$$x - \chi_u + \chi_v \in B \quad \text{or} \quad y + \chi_u - \chi_v \in B.$$

It may be noted that the properties (Q-EXC) and ($\mathrm{Q\text{-}EXC_w}$) are labeled (EXC) and ($\mathrm{EXC_w}$) in [19], respectively. The following properties for $B \subseteq \mathbf{Z}^V$ can be shown easily:

- ($\mathrm{Q\text{-}EXC_w}$) for B $\Leftrightarrow$ ($\mathrm{QM_w}$) for δ_B,
- (Q-EXC) for $B \Leftrightarrow$ (QM) for δ_B,
- (B-EXC) for $B \Leftrightarrow$ (SSQM) for $\delta_B \Leftrightarrow$ ($\mathrm{SSQM_w}$) for δ_B.

We show some examples of quasi M-convex functions below.

Example 3.2. Let $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. We define $f : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$\mathrm{dom}\, f = \{(x_1, x_2) \in \mathbf{Z}^2 \mid x_1 + x_2 = 0\},$$

$$f(x_1, x_2) = \varphi(x_1) \quad ((x_1, x_2) \in \mathrm{dom}\, f). \tag{3.3}$$

By Theorem 2.2, f satisfies (QM) (or ($\mathrm{QM_w}$)) if and only if φ is quasiconvex, and f satisfies (SSQM) (or ($\mathrm{SSQM_w}$)) if and only if φ is semistrictly quasiconvex.

Example 3.3. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be an M-convex function, and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a nondecreasing function. We define the function $\tilde{f} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\mathrm{dom}\, \tilde{f} = \mathrm{dom}\, f, \quad \tilde{f}(x) = \varphi(f(x)) \quad (x \in \mathrm{dom}\, \tilde{f}). \tag{3.4}$$

Then, $\tilde{f}$ satisfies (QM). Furthermore, if φ is strictly increasing, then $\tilde{f}$ satisfies (SSQM).

Example 3.4. Let $B \subseteq \mathbf{Z}^V$ be an M-convex set, $p \in \mathbf{R}^V$, and $\alpha \in \mathbf{R}$. Then, the set $S = \{x \in B \mid \langle p, x \rangle \leqslant \alpha\}$ satisfies (Q-EXC). Moreover, the function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f = S$ defined by $f(x) = \langle p, x \rangle$ $(x \in S)$ satisfies (SSQM).

Remark 3.5. The concept of (semistrict) quasi M-convexity can be naturally extended to functions $f : S \to T$ with $S \subseteq \mathbf{Z}^V$ and a totally ordered set T with total order $\preccurlyeq$. For example, the property (SSQM) is rewritten for such functions as follows:

$\forall x, y \in S$, $\forall u \in \mathrm{supp}^+(x - y)$, $\exists v \in \mathrm{supp}^-(x - y)$:

(i) if either $x - \chi_u + \chi_v \notin S$, or $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \succcurlyeq f(x)$, then $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \preccurlyeq f(y)$, and

(ii) if either $y + \chi_u - \chi_v \notin S$, or $y + \chi_u - \chi_v \in S$ and $f(y + \chi_u - \chi_v) \succcurlyeq f(y)$, then $x - \chi_u + \chi_v \in S$ and $f(x - \chi_u + \chi_v) \preccurlyeq f(x)$.

It is easy to see that the properties of (semistrictly) quasi M-convex functions shown in Sections 3 and 4.1 still hold true. For simplicity and convenience, however, we assume in this paper that the codomain of a function is $\mathbf{R} \cup \{+\infty\}$.

Example 3.6. Suppose $V = \{1, 2, \ldots, n\}$ $(n \geqslant 1)$. Let $a : V \to \mathbf{Z} \cup \{-\infty\}$, $b : V \to \mathbf{Z} \cup \{+\infty\}$, and $\alpha \in \mathbf{Z}$ satisfy $a(i) \leqslant b(i)$ $(i \in V)$ and $\sum_{i \in V} a(i) \leqslant \alpha \leqslant \sum_{i \in V} b(i)$. Let $f_i : [a(i), b(i)] \to \mathbf{R}$ $(i \in V)$ be a semistrictly quasiconvex function. We put $B = \{x \in [a, b] \mid x(V) = \alpha\}$ and define $f : B \to \mathbf{R}^V$ by $f(x) = (f_i(x(i)) \mid i \in V)$ $(x \in B)$, where the total order $\preccurlyeq$ on the codomain $\mathbf{R}^V$ of f is defined by the lexicographic order. Then, f satisfies (SSQM) in the extended sense (see Remark 3.5).

Proof. Let $x, y \in B$ be distinct vectors. Also, let $u \in \mathrm{supp}^+(x - y)$, $v \in \mathrm{supp}^-(x - y)$ be any elements, and w.l.o.g. assume that $u < v$. Then, we have $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$. If $f_u(x(u) - 1) < f_u(x(u))$ or $f_u(y(u) + 1) < f_u(y(u))$ holds, then we have $f(x - \chi_u + \chi_v) \prec f(x)$ or $f(y + \chi_u - \chi_v) \prec f(y)$. Otherwise, we have $f_u(x(u) - 1) = f_u(x(u))$ and $f_u(y(u) + 1) = f_u(y(u))$ by Theorem 2.2. If $f_v(x(v) + 1) < f_v(x(v))$ or $f_v(y(v) - 1) < f_v(y(v))$ holds, then we have $f(x - \chi_u + \chi_v) \prec f(x)$ or $f(y + \chi_u - \chi_v) \prec f(y)$. Otherwise, we have $f_v(x(v) + 1) = f_v(x(v))$ and $f_v(y(v) - 1) = f_v(y(v))$, from which follows $f(x - \chi_u + \chi_v) = f(x)$ and $f(y + \chi_u - \chi_v) = f(y)$. □

The relationship among various versions of quasi M-convex functions is summarized as follows.

Theorem 3.7. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then, we have*

$$\begin{array}{ccccc} \text{(M-EXC)} & \Rightarrow & \text{(SSQM)} & \Rightarrow & \text{(QM)} \\ \Updownarrow & & \Downarrow & & \Downarrow \\ (\text{M-EXC}_{\mathrm{w}}) & \Rightarrow & (\text{SSQM}_{\mathrm{w}}) & \Rightarrow & (\text{QM}_{\mathrm{w}}). \end{array}$$

The property ($\mathrm{QM_w}$) is equivalent to each of the following (seemingly) weaker conditions.

$$\max\{f(x), f(y)\} \geqslant \min_{u\in\mathrm{supp}^+(x-y)} \min_{v\in\mathrm{supp}^-(x-y)} \{f(x-\chi_u+\chi_v), f(y+\chi_u-\chi_v)\}$$
$$(\forall x, y \in \mathrm{dom}\, f \text{ with } x \neq y), \tag{3.5}$$

$$f(x) \geqslant \min_{u\in\mathrm{supp}^+(x-y)} \min_{v\in\mathrm{supp}^-(x-y)} f(x-\chi_u+\chi_v)$$
$$(\forall x, y \in \mathrm{dom}\, f \text{ with } x \neq y,\ f(x) \geqslant f(y)). \tag{3.6}$$

Theorem 3.8. *For* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, *we have* ($\mathrm{QM_w}$) $\Leftrightarrow$ (3.5) $\Leftrightarrow$ (3.6).

Proof. We prove "($\mathrm{QM_w}$) $\Rightarrow$ (3.6)" only. For $x, y \in \mathrm{dom}\, f$ with $f(x) \geqslant f(y)$, we show (3.6) by induction on the value $\|x-y\|_1$. We may assume $\|x-y\|_1 > 2$. Then, there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ with $\Delta f(x; v, u) \leqslant 0$ or $\Delta f(y; u, v) \leqslant 0$. If the latter holds, then the inductive hypothesis for x and $y' = y + \chi_u - \chi_v$ yields $\Delta f(x; v', u') \leqslant 0$ for some $u' \in \mathrm{supp}^+(x-y') \subseteq \mathrm{supp}^+(x-y)$ and $v' \in \mathrm{supp}^-(x-y') \subseteq \mathrm{supp}^-(x-y)$. □

3.2. Level sets of quasi M-convex functions

Level sets of quasi M-convex functions have quasi M-convexity. Furthermore, the weaker version of quasi M-convexity ($\mathrm{QM_w}$) for functions can be characterized by quasi M-convexity (Q-$\mathrm{EXC_w}$) of level sets.

Lemma 3.9 (Shioura [19]). *Let* $B \subseteq \mathbf{Z}^V$.

(i) *If B satisfies* (Q-$\mathrm{EXC_w}$), *then* $x(V) = y(V)$ *for all* $x, y \in B$.
(ii) (Q-$\mathrm{EXC_w}$) $\Leftrightarrow$ $\forall x, y \in B,\ x \neq y,\ \exists u \in \mathrm{supp}^+(x-y),\ \exists v \in \mathrm{supp}^-(x-y)$: $x - \chi_u + \chi_v \in B$.

Theorem 3.10. *A function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* ($\mathrm{QM_w}$) *if and only if the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (Q-$\mathrm{EXC_w}$) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *In particular, if f satisfies* ($\mathrm{QM_w}$), *then* $\mathrm{dom}\, f$ *and* $\arg\min f$ *satisfy* (Q-$\mathrm{EXC_w}$).

Proof. ["only if" part] Let $\alpha \in \mathbf{R} \cup \{+\infty\}$, and $x, y \in \mathrm{L}(f, \alpha)$ be vectors with $x \neq y$. Applying ($\mathrm{QM_w}$) to x and y, we have $\Delta f(x; v, u) \leqslant 0$ or $\Delta f(y; u, v) \leqslant 0$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. Therefore, we have $x - \chi_u + \chi_v \in \mathrm{L}(f, \alpha)$ or $y + \chi_u - \chi_v \in \mathrm{L}(f, \alpha)$.

["if" part] For any distinct $x, y \in \mathrm{dom}\, f$ with $f(x) \geqslant f(y)$, Lemma 3.9(ii) implies $x - \chi_u + \chi_v \in \mathrm{L}(f, f(x))$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$, i.e., $f(x - \chi_u + \chi_v) \leqslant f(x)$. □

Theorem 3.11. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$.

(i) *Assume* (QM) *for f. Then, the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (Q-EXC) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *In particular*, $\mathrm{dom}\, f$ *and* $\arg\min f$ *satisfy* (Q-EXC).
(ii) *If the level set* $\mathrm{L}(f, \alpha)$ *satisfies* (B-EXC) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$, *then f satisfies* (QM).
(iii) *If f satisfies* ($\mathrm{SSQM_w}$), *then* $\arg\min f$ *satisfies* (B-EXC).

An M-convex function can be characterized by quasi M-convexity of level sets of functions perturbed by linear functions. For $p \in \mathbf{R}^V$, the function $f[p] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is given by

$$f[p](x) = f(x) + \sum_{v \in V} p(v)x(v) \quad (x \in \mathbf{Z}^V). \tag{3.7}$$

Theorem 3.12 (Shioura [19, Theorem 1]). *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then,*

$$\begin{aligned} & f \ \textit{satisfies} \ \text{(M-EXC)} \\ & \quad \Leftrightarrow \ \forall p \in \mathbf{R}^V,\ \forall \alpha \in \mathbf{R} \cup \{+\infty\},\ \mathrm{L}(f[p], \alpha) \ \textit{satisfies} \ \text{(Q-EXC)} \\ & \quad \Leftrightarrow \ \forall p \in \mathbf{R}^V,\ \forall \alpha \in \mathbf{R} \cup \{+\infty\},\ \mathrm{L}(f[p], \alpha) \ \textit{satisfies} \ (\text{Q-EXC}_\text{w}). \end{aligned}$$

Combining Theorems 3.10 and 3.12, we see the following:

Corollary 3.13. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then, f satisfies* (M-EXC) $\Leftrightarrow \forall p \in \mathbf{R}^V$, $f[p]$ *satisfies* (QM) $\Leftrightarrow \forall p \in \mathbf{R}^V$, $f[p]$ *satisfies* ($\mathrm{QM_w}$).

3.3. Operations for quasi M-convex functions

The class of (semistrictly) quasi M-convex functions is closed under several fundamental operations. Proofs are clear from the definitions of (semistrictly) quasi M-convex functions.

Theorem 3.14. *Let* ($*\mathrm{QM}_*$) *denote one of* (QM), ($\mathrm{QM_w}$), (SSQM), *and* ($\mathrm{SSQM_w}$), *and* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with the property* ($*\mathrm{QM}_*$).

(i) *For any* $a \in \mathbf{Z}^V$ *and* $v > 0$, *the functions* $v f(a - x)$ *and* $v f(a + x)$ *satisfy* ($*\mathrm{QM}_*$) *as a function in* x.
(ii) *For any* $U \subseteq V$, *the function* $f_U : \mathbf{Z}^U \to \mathbf{R} \cup \{+\infty\}$ *defined by* $f_U(y) = f(y, \mathbf{0}_{V \setminus U})$ $(y \in \mathbf{Z}^U)$ *satisfies* ($*\mathrm{QM}_*$), *where* $\mathbf{0}_{V \setminus U} \in \mathbf{Z}^{V \setminus U}$ *denotes the zero vector*.
(iii) *For any* $a : V \to \mathbf{Z} \cup \{-\infty\}$ *and* $b : V \to \mathbf{Z} \cup \{+\infty\}$ *with* $a \leqslant b$, *the function* $f_a^b : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *defined by*

$$\mathrm{dom}\, f_a^b = \mathrm{dom}\, f \cap [a, b], \qquad f_a^b(x) = f(x) \quad (x \in \mathrm{dom}\, f_a^b) \tag{3.8}$$

satisfies ($*\mathrm{QM}_*$).

(iv) *For* $i=1,2$, *let* V_i *be a finite set, and assume* $(*\mathrm{QM}_*)$ *for* $f_i:\mathbf{Z}^{V_i}\to\mathbf{R}_{++}\cup\{+\infty\}$. *Then, the function* $f:\mathbf{Z}^{V_1}\times\mathbf{Z}^{V_2}\to\mathbf{R}_{++}\cup\{+\infty\}$ *defined by* $f(x_1,x_2)=f_1(x_1)f_2(x_2)$ $((x_1,x_2)\in\mathbf{Z}^{V_1}\times\mathbf{Z}^{V_2})$ *satisfies* $(*\mathrm{QM}_*)$.

Remark 3.15. The class of (semistrictly) quasi M-convex functions is not closed under addition; it is not closed under the addition of a linear function.

Theorem 3.16. *For* $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$ *and* $\varphi:\mathbf{R}\to\mathbf{R}\cup\{+\infty\}$, *define* $\tilde{f}:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$ *by* (3.4).

(i) *If f satisfies* (QM) (*resp.* ($\mathrm{QM_w}$)) *and* φ *is nondecreasing, then* $\tilde{f}$ *satisfies* (QM) (*resp.* ($\mathrm{QM_w}$)).
(ii) *If f satisfies* (SSQM) (*resp.* ($\mathrm{SSQM_w}$)) *and* φ *is strictly increasing, then* $\tilde{f}$ *satisfies* (SSQM) (*resp.* ($\mathrm{SSQM_w}$)).

Theorem 3.17. *Let* $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$ *and* $g:\mathbf{Z}^V\to\mathbf{R}\cup\{-\infty\}$ *be such that* $g(x)>0$ $(\forall x\in\mathrm{dom}\,f)$. *If the function* $f(\cdot)-\alpha g(\cdot)$ *satisfies* ($\mathrm{QM_w}$) *for all* $\alpha\in\mathbf{R}$, *then the function* $r:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$ *with* $\mathrm{dom}\,r=\mathrm{dom}\,f$ *given by* $r(x)=f(x)/g(x)$ $(x\in\mathrm{dom}\,r)$ *satisfies* ($\mathrm{QM_w}$).

Proof. The proof is immediate from Theorem 3.10. □

3.4. Characterization of quasi M-convexity by local exchange properties

M-convex sets and functions are known to be characterized by localized properties:

(B-EXC-loc) $\forall x,y\in\mathrm{dom}\,f$ with $\|x-y\|_1=4$, $\forall u\in\mathrm{supp}^+(x-y)$, $\exists v\in\mathrm{supp}^-(x-y)$: $x-\chi_u+\chi_v\in B$ and $y+\chi_u-\chi_v\in B$.

(M-EXC-loc) $\forall x,y\in\mathrm{dom}\,f$ with $\|x-y\|_1=4$, $\forall u\in\mathrm{supp}^+(x-y)$, $\exists v\in\mathrm{supp}^-(x-y)$ satisfying (1.1).

Theorem 3.18 (Murota [10, Theorem 3.1], Shioura [19, Theorem 2]). *Let* $B\subseteq\mathbf{Z}^V$ *and* $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$, *and assume* ($\mathrm{Q\text{-}EXC_w}$) *for B and* $\mathrm{dom}\,f$. *Then,*
(i) (B-EXC) $\Leftrightarrow$ (B-EXC-loc), *and* (ii) (M-EXC) $\Leftrightarrow$ (M-EXC-loc).

We show that semistrict quasi M-convexities can be characterized also by the localized versions of (SSQM) and ($\mathrm{SSQM_w}$):

(SSQM-loc) $\forall x,y\in\mathrm{dom}\,f$ with $\|x-y\|_1=4$, $\forall u\in\mathrm{supp}^+(x-y)$, $\exists v\in\mathrm{supp}^-(x-y)$ satisfying (3.2).

($\mathrm{SSQM_w}$-loc) $\forall x,y\in\mathrm{dom}\,f$ with $\|x-y\|_1=4$, $\exists u\in\mathrm{supp}^+(x-y)$, $\exists v\in\mathrm{supp}^-(x-y)$ satisfying (3.2).

Theorem 3.19. *Let* $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$, *and assume* ($\mathrm{Q\text{-}EXC_w}$) *for* $\mathrm{dom}\,f$. *Then,* (i) (SSQM) $\Leftrightarrow$ (SSQM-loc), *and* (ii) ($\mathrm{SSQM_w}$) $\Leftrightarrow$ ($\mathrm{SSQM_w}$-loc).

Proof. For both (i) and (ii), the "$\Rightarrow$" parts are obvious.

["$\Leftarrow$" part of (i)] Assume, to the contrary, that (SSQM) does not hold for some $x, y \in \mathrm{dom}\, f$ and $u_* \in \mathrm{supp}^+(x-y)$. We also assume that (x, y) minimizes the value $\|x-y\|_1$ of all such pairs. Note that $\|x-y\|_1 \geqslant 6$ and $x(V)=y(V)$ by Lemma 3.9(i).

Claim 1. *There exists $u_0 \in \mathrm{supp}^+(x-y)$ such that $y+\chi_{u_0}-\chi_v \in \mathrm{dom}\, f$ for some $v \in \mathrm{supp}^-(x-y)$. Moreover, if $x(u_*)-y(u_*)=1$ then we can assume $u_0 \neq u_*$.*

Proof. From Lemma 3.9(ii), there exist some $u_1 \in \mathrm{supp}^+(x-y)$ and $v_1 \in \mathrm{supp}^-(x-y)$ with $y_1 = y+\chi_{u_1}-\chi_{v_1} \in \mathrm{dom}\, f$, which proves the former part of Claim 1. In the following, we assume $x(u_*)-y(u_*)=1$ and show the latter part of Claim 1. If $u_1 \neq u_*$ then we are done. Thus, we assume $u_1 = u_*$. Since $\|x-y_1\|_1 \geqslant 4$, we can again apply Lemma 3.9(ii) to y_1 and x to obtain $u_2 \in \mathrm{supp}^+(x-y_1) = \mathrm{supp}^+(x-y)\setminus\{u_*\}$ and $v_2 \in \mathrm{supp}^-(x-y_1) \subseteq \mathrm{supp}^-(x-y)$ with $y_2 = y_1+\chi_{u_2}-\chi_{v_2} \in \mathrm{dom}\, f$. Then, we apply (SSQM-loc) to y_2, y, and $u_* \in \mathrm{supp}^+(y_2-y)$ to obtain some $v \in \mathrm{supp}^-(y_2-y)=\{v_1, v_2\}$ such that if $\Delta f(y; u_*, v) \geqslant 0$ then $\Delta f(y_2; v, u_*) \leqslant 0$. By the choice of x and y we have $\Delta f(y; u_*, v) \geqslant 0$, from which follows $\Delta f(y_2; v, u_*) \leqslant 0$. Hence, $y_2+\chi_v-\chi_{u_*} = y+\chi_{u_2}-\chi_{v'} \in \mathrm{dom}\, f$ for some $v' \in \{v_1, v_2\}$. $\square$

We can divide the set $\mathrm{supp}^-(x-y)$ into three sets $S^-_{\gg}$, $S^-_{>=}$, and $S^-_{=>}$, where

$$S^-_{\gg} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) > 0, \Delta f(y; u_*, v) > 0\},$$

$$S^-_{>=} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) > 0, \Delta f(y; u_*, v) = 0\},$$

$$S^-_{=>} = \{v \in \mathrm{supp}^-(x-y) \mid \Delta f(x; v, u_*) = 0, \Delta f(y; u_*, v) > 0\}.$$

Then, we choose $v_0 \in \mathrm{supp}^-(x-y)$ as follows: if

$$\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>=}\} < \min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{\gg} \cup S^-_{=>}\},$$

then let $v_0 \in \arg\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{>=}\}$, and otherwise let $v_0 \in \arg\min\{f(y+\chi_{u_0}-\chi_v) \mid v \in S^-_{\gg} \cup S^-_{=>}\}$. Put $y' = y+\chi_{u_0}-\chi_{v_0}$. Then, $y' \in \mathrm{dom}\, f$ by Claim 1.

Claim 2. $\Delta f(y'; u_*, v) \geqslant 0$ *for* $v \in \mathrm{supp}^-(x-y')$. *In particular, if $v \in \mathrm{supp}^-(x-y') \cap S^-_{=>}$, then $\Delta f(y'; u_*, v) > 0$.*

Proof. For $v \in \mathrm{supp}^-(x-y')$, put $y'' = y'+\chi_{u_*}-\chi_v = y+\chi_{u_0}+\chi_{u_*}-\chi_{v_0}-\chi_v$. We may assume $y'' \in \mathrm{dom}\, f$. Applying (SSQM-loc) to y'', y, and $u_* \in \mathrm{supp}^+(y''-y)$, we have

$$\begin{aligned} &\Delta f(y''; v', u_*) \geqslant 0 \;\Rightarrow\; \Delta f(y; u_*, v') \leqslant 0, \\ &\Delta f(y; u_*, v') \geqslant 0 \;\Rightarrow\; \Delta f(y''; v', u_*) \leqslant 0 \end{aligned} \tag{3.9}$$

for some $v' \in \{v_0, v\}$. Since $\Delta f(y; u_*, v') \geqslant 0$, (3.9) implies

$$\begin{aligned} f(y'') &\geqslant f(y''+\chi_{v'}-\chi_{u_*}) = f(y+\chi_{u_0}-\chi_{v_0}-\chi_v+\chi_{v'}) \\ &\geqslant f(y+\chi_{u_0}-\chi_{v_0}) = f(y'). \end{aligned} \tag{3.10}$$

This proves the former part of Claim 2.

Next, we assume $v \in S^-_{=>}$. It suffices to show that one of the two inequalities in (3.10) holds with strict inequality. If $\Delta f(y; u_*, v') > 0$, then (3.9) implies $f(y'') > f(y'' + \chi_{v'} - \chi_{u_*})$. Hence, we assume $\Delta f(y; u_*, v') = 0$, implying $v' = v_0 \in S^-_{>=}$ since $v \in S^-_{=>}$. Due to the choice of v_0, we have $f(y') < f(y + \chi_{u_0} - \chi_v) = f(y + \chi_{u_0} - \chi_{v_0} - \chi_v + \chi_{v'})$. □

Since $u_* \in \mathrm{supp}^+(x - y')$ and $\|x - y'\|_1 < \|x - y\|_1$, Claim 2 contradicts the choice of x and y.

["⇐" part of (ii)] We show ($\mathrm{SSQM_w}$) for $x, y \in \mathrm{dom}\, f$ by induction on the value $\|x - y\|_1$. We may assume that $k = \|x - y\|_1/2 > 2$ and

$$\Delta f(x; v, u) \geqslant 0, \quad \Delta f(y; u, v) \geqslant 0$$
$$(\forall u \in \mathrm{supp}^+(x - y), \forall v \in \mathrm{supp}^-(x - y)). \tag{3.11}$$

We are to show $\Delta f(x; v, u) = \Delta f(y; u, v) = 0$ for some $u \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$.

Define $x_i \in \mathbf{Z}^V$ $(i = 0, 1, \ldots, k)$ iteratively by $x_0 = x$ and $x_i = x_{i-1} - \chi_{u_i} + \chi_{v_i}$ with $u_i \in \mathrm{supp}^+(x_{i-1} - y)$ and $v_i \in \mathrm{supp}^-(x_{i-1} - y)$ such that $f(x_i) \leqslant f(x_{i-1} - \chi_u + \chi_v)$ for all $u \in \mathrm{supp}^+(x_{i-1} - y)$ and $v \in \mathrm{supp}^-(x_{i-1} - y)$. Note that $x_k = y$.

Claim 1. $f(x_0)\ (= f(x)) = f(x_1) = f(x_2) = \cdots = f(x_k)\ (= f(y))$.

Proof. We first prove $f(x_0) \leqslant f(x_1) \leqslant f(x_2) \leqslant \cdots \leqslant f(x_k)$. From ($\mathrm{Q\text{-}EXC_w}$) for $\mathrm{dom}\, f$ follows $x_i \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f$ $(i = 1, \ldots, k)$. We show the inequality $f(x_{i+1}) \geqslant f(x_i)$ by induction on i. From (3.11) follows $f(x_1) \geqslant f(x_0)$. We then suppose $i \geqslant 1$. Since $\|x_{i-1} - x_{i+1}\|_1 = 4$, we can apply ($\mathrm{SSQM_w}$-loc) to x_{i-1} and x_{i+1} to obtain some $u \in \mathrm{supp}^+(x_{i-1} - x_{i+1})$ and $v \in \mathrm{supp}^-(x_{i-1} - x_{i+1})$ such that if $\Delta f(x_{i-1}; v, u) \geqslant 0$ then $\Delta f(x_{i+1}; u, v) \leqslant 0$. By the inductive hypothesis and the choice of x_i, we have $\Delta f(x_{i-1}; v, u) \geqslant f(x_i) - f(x_{i-1}) \geqslant 0$. Hence, $f(x_{i+1}) \geqslant f(x_{i+1} + \chi_u - \chi_v) \geqslant f(x_i)$ follows. We can show $f(x) \geqslant f(y)$ in a similar way. Thus, we have the claim. □

Claim 2. $f(x) = f(y) = \min\{f(x') \mid x' \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f\}$.

Proof. The proof is similar to that for Claim 1. □

Claim 3. $S_0 = \arg\min\{f(x') \mid x' \in [x \wedge y, x \vee y] \cap \mathrm{dom}\, f\}$ *satisfies* (B-EXC).

Proof. From Theorem 3.18(i), we show ($\mathrm{Q\text{-}EXC_w}$) and (B-EXC-loc) for S_0.

Let $\tilde{x}, \tilde{y} \in S_0$ be distinct vectors, and firstly assume $\|\tilde{x} - \tilde{y}\|_1 < \|x - y\|_1$. Then, we can apply ($\mathrm{SSQM_w}$) to $\tilde{x}$ and $\tilde{y}$ by the inductive hypothesis. Due to the definition of S_0, we have $\Delta f(\tilde{x}; v, u) = \Delta f(\tilde{y}; u, v) = 0$ for some $u \in \mathrm{supp}^+(\tilde{x} - \tilde{y})$ and $v \in \mathrm{supp}^-(\tilde{x} - \tilde{y})$, i.e., ($\mathrm{Q\text{-}EXC_w}$) holds for $\tilde{x}$ and $\tilde{y}$. This fact also shows (B-EXC-loc) for S_0.

We then assume $\|\tilde{x} - \tilde{y}\|_1 = \|x - y\|_1$. If $\{\tilde{x}, \tilde{y}\} = \{x, y\}$, then ($\mathrm{Q\text{-}EXC_w}$) follows from Claim 1. Otherwise, we have $\|\tilde{x} - x\|_1 < \|\tilde{x} - \tilde{y}\|_1$, $\mathrm{supp}^+(\tilde{x} - x) \subseteq \mathrm{supp}^+(\tilde{x} - \tilde{y})$, and $\mathrm{supp}^-(\tilde{x}-x) \subseteq \mathrm{supp}^-(\tilde{x}-\tilde{y})$. Hence, ($\mathrm{SSQM_w}$) for $\tilde{x}$ and x implies ($\mathrm{Q\text{-}EXC_w}$) for $\tilde{x}$ and $\tilde{y}$. $\square$

Applying (B-EXC) to x, y, we have $x-\chi_u+\chi_v \in S_0$ and $y+\chi_u-\chi_v \in S_0$ for some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$. Hence follows $\Delta f(x;v,u)=\Delta f(y;u,v)=0$. $\square$

4. Minimization of quasi M-convex functions

In this section, we use the following weaker properties than (SSQM) and ($\mathrm{SSQM_w}$):

($\mathrm{SSQM}^{\neq}$) $\forall x, y \in \mathrm{dom}\, f$ with $f(x) \neq f(y)$, $\forall u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (3.2).

($\mathrm{SSQM}^{\neq}_{\mathrm{w}}$) $\forall x, y \in \mathrm{dom}\, f$ with $f(x) \neq f(y)$, $\exists u \in \mathrm{supp}^+(x-y)$, $\exists v \in \mathrm{supp}^-(x-y)$ satisfying (3.2).

The property ($\mathrm{SSQM}^{\neq}_{\mathrm{w}}$) is equivalent to each of the following two conditions, which can be shown similarly to that for Theorem 3.8. Condition (4.2) is also considered in [16].

$$\max\{f(x), f(y)\} > \min_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} \{f(x-\chi_u+\chi_v), f(y+\chi_u-\chi_v)\}$$
$$(\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) \neq f(y)), \tag{4.1}$$

$$f(x) > \min_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} f(x-\chi_u+\chi_v)$$
$$(\forall x, y \in \mathrm{dom}\, f \text{ with } f(x) > f(y)). \tag{4.2}$$

Theorem 4.1. *For* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, ($\mathrm{SSQM}^{\neq}_{\mathrm{w}}$) $\Leftrightarrow$ (4.1) $\Leftrightarrow$ (4.2).

4.1. Properties of minimizers of quasi M-convex functions

Global minimality of quasi M-convex functions is characterized by local minimality.

Theorem 4.2. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $x \in \mathrm{dom}\, f$.

(i) *Assume* ($\mathrm{QM_w}$) *for f. Then,*

$$\Delta f(x;v,u) > 0 \ (\forall u, v \in V, u \neq v) \quad \Leftrightarrow \quad f(x) < f(y) \ (\forall y \in \mathbf{Z}^V \setminus \{x\}).$$

(ii) *Assume* ($\mathrm{SSQM}^{\neq}_{\mathrm{w}}$) *for f. Then,*

$$\Delta f(x;v,u) \geqslant 0 \ (\forall u, v \in V) \quad \Leftrightarrow \quad f(x) \leqslant f(y) \ (\forall y \in \mathbf{Z}^V).$$

Proof. We show the "$\Rightarrow$" part of (ii). Let $y \in \mathrm{dom}\, f$ be with $f(y) < f(x)$. By Theorem 4.1, there exist some $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that

$\Delta f(x; v, u) < 0$. The "$\Leftarrow$" part of (ii) is obvious, and we can show (i) similarly to that of (ii) by Theorem 3.8. □

If f satisfies (SSQM$^{\neq}$), then any vector in $\mathrm{dom}\, f$ can be easily separated from some minimizer of f (cf. [18, Theorem 2.2, Corollary 2.3]). This property will be used as a basis of the domain reduction method in Section 4.2.

Theorem 4.3. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with* (SSQM$^{\neq}$), *and* $x \in \mathrm{dom}\, f$. *Assume* $\arg\min f \neq \emptyset$.

(i) *For* $v \in V$, *let* $u \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{s \in V} f(x - \chi_s + \chi_v)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(u) \leqslant x(u) - 1 + \chi_v(u)$.
(ii) *For* $u \in V$, *let* $v \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{t \in V} f(x - \chi_u + \chi_t)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(v) \geqslant x(v) - \chi_u(v) + 1$.
(iii) *Assume* $x \notin \arg\min f$, *and let* $u, v \in V$ *be such that* $f(x - \chi_u + \chi_v) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$. *Then, there exists* $x_* \in \arg\min f$ *with* $x_*(u) \leqslant x(u) - 1$ *and* $x_*(v) \geqslant x(v) + 1$.

Proof. (i) Put $x' = x - \chi_u + \chi_v$. Assume, to the contrary, that there is no $x_* \in \arg\min f$ with $x_*(u) \leqslant x'(u)$. Let $x_* \in \arg\min f$ minimize $x_*(u)$. Then, we have $x_*(u) > x'(u)$. Since $f(x_*) \neq f(x')$, we can apply (SSQM$^{\neq}$) to x_*, x', and u to obtain some $w \in \mathrm{supp}^-(x_* - x')$ such that if $\Delta f(x_*; w, u) > 0$ then $\Delta f(x'; u, w) < 0$. Due to the choice of x_*, we have $\Delta f(x_*; w, u) > 0$. Hence, $f(x') > f(x' + \chi_u - \chi_w) = f(x - \chi_w + \chi_v)$ holds, a contradiction to the definition of $u \in V$.

(ii) The proof is similar to that for (i).

(iii) Put $x' = x - \chi_u + \chi_v$ ($\neq x$). By (i), there exists some $x_* \in \arg\min f$ such that $x_*(u) \leqslant x'(u)$, and we suppose that x_* maximizes $x_*(v)$ among all such vectors. To the contrary assume $x_*(v) < x'(v)$. Since $f(x_*) \neq f(x')$, we can apply (SSQM$^{\neq}$) to x', x_*, and v to obtain some $w \in \mathrm{supp}^-(x' - x_*)$ satisfying at least one of (a) $\Delta f(x'; w, v) < 0$, (b) $\Delta f(x_*; v, w) < 0$, and (c) $\Delta f(x'; w, v) = \Delta f(x_*; v, w) = 0$. Due to the choice of $u, v \in V$, we have $\Delta f(x'; w, v) \geqslant 0$ since $x' - \chi_v + \chi_w = x - \chi_u + \chi_w$. We also have $\Delta f(x_*; v, w) \geqslant 0$ since $x_* \in \arg\min f$. Therefore, we have (c), which implies $x_* + \chi_v - \chi_w \in \arg\min f$, a contradiction to the choice of x_*. □

We apply the scaling technique to the minimization of quasi M-convex functions in Section 4.2. Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a semistrictly quasi M-convex function and α be any positive integer. Let $x_\alpha \in \mathrm{dom}\, f$ be an approximate minimum of f in the sense that x_α satisfies

$$f(x_\alpha) \leqslant f(x_\alpha + \alpha(\chi_v - \chi_u)) \quad (\forall u, v \in V). \tag{4.3}$$

The following is a proximity theorem showing that a global minimum of a semistrictly M-convex function exists in the neighborhood of x_α. This generalizes a proximity theorem for M-convex functions in [9]. This proximity theorem also has a similar flavor to the one by Hochbaum for the nonlinear resource allocation problem [5, Theorem 4.1].

Theorem 4.4. *Let* $f:\mathbf{Z}^V\to\mathbf{R}\cup\{+\infty\}$ *be a function with* (SSQM$^{\neq}$) *and* $\alpha\in\mathbf{Z}_{++}$. *Suppose that* $x_\alpha\in\mathrm{dom}\,f$ *satisfies* (4.3). *Then*, $\arg\min f\neq\emptyset$ *and there exists some* $x_*\in\arg\min f$ *with*

$$|x_\alpha(v)-x_*(v)|\leqslant(n-1)(\alpha-1)\quad(v\in V).\tag{4.4}$$

Proof. It suffices to show that for any $\gamma\in\mathbf{R}$ with $\gamma>\inf f$, there exists some $x_*\in\mathrm{dom}\,f$ satisfying $f(x_*)\leqslant\gamma$ and (4.4). Let $x_*\in\mathrm{dom}\,f$ satisfy $f(x_*)\leqslant\gamma$, and suppose that x_* minimizes the value $\|x_*-x_\alpha\|_1$ among all such vectors. In the following, we fix $v\in V$ and prove $x_\alpha(v)-x_*(v)\leqslant(n-1)(\alpha-1)$. The inequality $x_*(v)-x_\alpha(v)\leqslant(n-1)(\alpha-1)$ can be shown similarly.

We may assume $x_\alpha(v)>x_*(v)$. Put

$$S=\Bigg\{x_\alpha-\lambda\chi_v+\sum_{w\in\mathrm{supp}^-(x_\alpha-x_*)}\mu_w\chi_w\in\mathrm{dom}\,f\ \Big|\ 0\leqslant\lambda\leqslant x_\alpha(v)-x_*(v),\ \lambda=\sum\{\mu_w\mid w\in\mathrm{supp}^-(x_\alpha-x_*)\},\ 0\leqslant\mu_w\leqslant x_*(w)-x_\alpha(w)\ (w\in\mathrm{supp}^-(x_\alpha-x_*))\Bigg\}$$

Claim 1. *For* $y\in\arg\min\{f(y')\mid y'\in S\}$, *we have* $y(v)=x_*(v)$.

Proof. Assume, to the contrary, that $y(v)>x_*(v)$. Since $\|y-x_\alpha\|_1<\|x_*-x_\alpha\|_1$, we have $f(y)>f(x_*)$. By (SSQM$^{\neq}$) applied to y, x_*, and $v\in\mathrm{supp}^+(y-x_*)\subseteq\mathrm{supp}^+(x_\alpha-x_*)$, we have some $w\in\mathrm{supp}^-(y-x_*)\subseteq\mathrm{supp}^-(x_\alpha-x_*)$ such that if $\Delta f(x_*;v,w)>0$ then $\Delta f(y;w,v)<0$. By the choice of x_*, we have $\Delta f(x_*;v,w)>0$ since $\|(x_*+\chi_v-\chi_w)-x_\alpha\|_1<\|x_*-x_\alpha\|_1$. Therefore, $f(y-\chi_v+\chi_w)<f(y)$, which is a contradiction since $y-\chi_v+\chi_w\in S$. □

Let $\tilde y=x_\alpha-\tilde\lambda\chi_v+\sum\{\tilde\mu_w\chi_w\mid w\in\mathrm{supp}^-(x_\alpha-x_*)\}\in\arg\min\{f(y')\mid y'\in S\}$.

Claim 2. *For any* $w\in\mathrm{supp}^-(x_\alpha-x_*)$ *with* $\tilde\mu_w>0$ *and* $\mu\in[0,\tilde\mu_w-1]$, *we have* $x_\alpha-(\mu+1)(\chi_v-\chi_w)\in\mathrm{dom}\,f$ *and* $f(x_\alpha-(\mu+1)(\chi_v-\chi_w))<f(x_\alpha-\mu(\chi_v-\chi_w))$.

Proof. For $\mu\in[0,\tilde\mu_w-1]$, put $x'=x_\alpha-\mu(\chi_v-\chi_w)$ and suppose $x'\in\mathrm{dom}\,f$. Claim 1 yields $f(x')>f(\tilde y)$ since $x'\in S$ and $x'(v)>x_*(v)$. Since $\mathrm{supp}^-(\tilde y-x')=\{v\}$, (SSQM$^{\neq}$) applied to $\tilde y$, x', and $w\in\mathrm{supp}^+(\tilde y-x')$ implies that if $\Delta f(\tilde y;v,w)>0$ then $\Delta f(x';w,v)<0$. By Claim 1, we have $\Delta f(\tilde y;v,w)>0$, from which the claim follows. □

Claim 2 and (4.3) imply $\tilde\mu_w\leqslant\alpha-1$ for $w\in\mathrm{supp}^-(x_\alpha-x_*)$. Thus,

$$x_\alpha(v)-x_*(v)=x_\alpha(v)-\tilde y(v)=\tilde\lambda=\sum_{w\in\mathrm{supp}^-(x_\alpha-x_*)}\tilde\mu_w\leqslant(n-1)(\alpha-1).\qquad\square$$

4.2. Algorithms

Let $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function such that $\mathrm{dom}\, f$ is a nonempty bounded set, and put $L = \max\{|x(v) - y(v)| \mid x, y \in \mathrm{dom}\, f,\ v \in V\}$. Throughout this section, we assume that an initial vector $x_0 \in \mathrm{dom}\, f$ is given a priori, and that we have an oracle for computing a function value of f in unit time.

Remark 4.5. It is difficult to find a vector in $\mathrm{dom}\, f$ efficiently even if $\mathrm{dom}\, f$ is a bounded set. A simple example is a function $f : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f = \{\alpha\}$ for some $\alpha \in \mathbf{Z}$, for which we have no efficient algorithm for finding such α.

Assume (SSQM$^{\neq}_{\mathrm{w}}$) for f. Then, Theorem 4.2 immediately leads to the following algorithm.

Algorithm DESCENT_M
Step 0: Let $x := x_0$ $(\in \mathrm{dom}\, f)$.
Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]
Step 2: Find $u, v \in V$ with $f(x - \chi_u + \chi_v) < f(x)$.
Step 3: Set $x := x - \chi_u + \chi_v$. Go to Step 1.

Algorithm DESCENT_M terminates in at most $|\mathrm{dom}\, f| \leqslant (L+1)^{n-1}$ iterations since it generates a distinct x in each iteration.

To the end of this section we assume (SSQM$^{\neq}$) for f. Based on Theorem 4.4, we apply the scaling technique to Algorithm DESCENT_M to obtain a faster algorithm.

Algorithm SCALING_DESCENT_M
Step 0: Let $x := x_0$ $(\in \mathrm{dom}\, f)$. Put $\alpha := 2^{\lceil \log_2 L \rceil}$, $B := \mathrm{dom}\, f$.
Step 1:
Step 1-1: If $f(x) = \min\{f(x - \alpha(\chi_s - \chi_t)) \mid s, t \in V,\ x - \alpha(\chi_s - \chi_t) \in B\}$, then go to Step 2.
Step 1-2: Find $u, v \in V$ with $x - \alpha(\chi_u - \chi_v) \in B$ satisfying $f(x - \alpha(\chi_u - \chi_v)) < f(x)$.
Step 1-3: Set $x := x - \alpha(\chi_u - \chi_v)$. Go to Step 1-1.
Step 2: If $\alpha = 1$ then stop. [x is a minimizer of f.]
Step 3: Put $B := B \cap \{y \in \mathbf{Z}^V \mid |y(v) - x(v)| \leqslant (n-1)(\alpha - 1)\ (v \in V)\}$ and $\alpha := \alpha/2$. Go to Step 1.

The number of scaling phases is $\lceil \log_2 L \rceil$, and each scaling phase terminates in $(4n)^{n-1}$ iterations since we have $\max\{|x(v) - y(v)| \mid x, y \in B\} < 4n\alpha$ $(v \in V)$ in Step 1. Therefore, Algorithm SCALING_DESCENT_M runs in $(4n)^{n-1} \lceil \log_2 L \rceil$ iterations.

We then propose another elaboration of Algorithm DESCENT_M. Note that the algorithm STEEPEST_DESCENT_M reduces the set B iteratively in Step 3 by exploiting Theorem 4.3(iii).

Algorithm STEEPEST_DESCENT_M
Step 0: Let $x := x_0$ $(\in \mathrm{dom}\, f)$. Set $B := \mathrm{dom}\, f$.
Step 1: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]
Step 2: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying

$$f(x - \chi_u + \chi_v) = \min\{f(x - \chi_s + \chi_t) \mid s, t \in V,\ x - \chi_s + \chi_t \in B\}. \tag{4.5}$$

Step 3: Set $x := x - \chi_u + \chi_v$ and

$$B := B \cap \{y \in \mathbf{Z}^V \mid y(u) \leqslant x(u) - 1,\ \ y(v) \geqslant x(v) + 1\}. \tag{4.6}$$

Go to Step 1.

By Theorem 4.3(iii), the set B always contains a minimizer of f. Hence, STEEPEST_DESCENT_M finds a minimizer of f. To analyze the number of iterations, we consider the value $\sum_{w \in V} \{u_B(w) - l_B(w)\}$, where $l_B(w) = \min_{y \in B} y(w)$ and $u_B(w) = \max_{y \in B} y(w)$ $(w \in V)$. This value is bounded by nL and decreases at least by two in each iteration. Therefore, STEEPEST_DESCENT_M terminates in $\mathrm{O}(nL)$ iterations. In particular, if $\mathrm{dom}\, f \subseteq \{0,1\}^V$ then the number of iterations is $\mathrm{O}(n^2)$.

It is shown in [18] that the minimization of an M-convex function can be done in polynomial time by the domain reduction method explained below. We show that the domain reduction method also works for the minimization of a function with (SSQM$^{\neq}$) if its effective domain is a bounded M-convex set.

Given a bounded M-convex set $B \subseteq \mathbf{Z}^V$, we define $N_B = \{y \in B \mid l'_B \leqslant y \leqslant u'_B\}$, where for $w \in V$ we put

$$l'_B(w) = \left\lfloor \frac{n-1}{n}\, l_B(w) + \frac{1}{n}\, u_B(w) \right\rfloor, \quad u'_B(w) = \left\lceil \frac{1}{n}\, l_B(w) + \frac{n-1}{n}\, u_B(w) \right\rceil.$$

Then, N_B is a (nonempty) M-convex set [18, Theorem 2.4]. The next algorithm maintains an M-convex set B containing a minimizer of f. It reduces B iteratively by exploiting Theorem 4.3(iii) and finally finds a minimizer.

Algorithm DOMAIN_REDUCTION
Step 0: Set $B := \mathrm{dom}\, f$.
Step 1: Find a vector $x \in N_B$.
Step 2: If $f(x) = \min_{s,t \in V} f(x - \chi_s + \chi_t)$ then stop. [x is a minimizer of f.]
Step 3: Find $u, v \in V$ with $x - \chi_u + \chi_v \in B$ satisfying (4.5).
Step 4: Set B by (4.6). Go to Step 1.

We analyze the time complexity of DOMAIN_REDUCTION. Denote by B_i the set B in the ith iteration, and let $l_i(w) = l_{B_i}(w)$, $u_i(w) = u_{B_i}(w)$ $(w \in V)$. It is clear that $u_i(w) - l_i(w)$ is nonincreasing w.r.t. i. Moreover, we have $u_{i+1}(w) - l_{i+1}(w) < (1 - 1/n)\{u_i(w) - l_i(w)\}$ for $w \in \{u, v\}$, where $u, v \in V$ are the elements found in Step 3 [18, Lemma 3.1], which implies that the algorithm terminates in $\mathrm{O}(n^2 \log L)$ iterations.

Steps 2–4 can be done in $\mathrm{O}(n^2)$ time. In Step 1, we need to evaluate the exchange capacity $\mathrm{O}(n^2)$ times. For $x \in \mathrm{dom}\, f$ and $u, v \in V$, the exchange capacity associated with x, v and u is defined as $\tilde{c}(x, v, u) = \max\{\alpha \in \mathbf{Z} \mid x + \alpha(\chi_v - \chi_u) \in \mathrm{dom}\, f\}$, and can be computed in $\mathrm{O}(\log L)$ time by binary search. Hence, Step 1 requires $\mathrm{O}(n^2 \log L)$ time. See [18] for details of the analysis.

Theorem 4.6. *Suppose that* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (SSQM$^{\neq}$) *and that* $\mathrm{dom}\, f$ *is a bounded M-convex set. If a vector in* $\mathrm{dom}\, f$ *is given, Algorithm* DOMAIN_REDUCTION *finds a minimizer of* f *in* $\mathrm{O}(n^4 (\log L)^2)$ *time.*

Table 2
Possible sign patterns of $g(p \wedge q) - g(p)$ and $g(p \vee q) - g(q)$ in submodular inequality

$g(p \wedge q) - g(p) \setminus g(p \vee q) - g(q)$	$-$	0	$+$
$-$	○	○	○
0	○	○	×
$+$	○	×	×

○ ··· possible, × ··· impossible.

5. Quasi L-convex and submodular functions

We first review the concept of L-convexity for sets and functions. A set $D \subseteq \mathbf{Z}^V$ is called *L-convex* if $D \neq \emptyset$ and it satisfies (DL) and (TRS):

(DL) $p, q \in D \Rightarrow p \wedge q,\ p \vee q \in D$, (TRS) $p \in D, \lambda \in \mathbf{Z} \Rightarrow p + \lambda \mathbf{1} \in D$.

A function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is called *L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (SBM) and (TRF) (see Introduction for the properties (SBM) and (TRF)).

5.1. Definition of quasi L-convex and submodular functions

To extend the concept of L-convexity to quasi L-convexity, we relax the submodularity condition (SBM) while keeping in mind the possible sign patterns of the values $g(p \wedge q) - g(p)$ and $g(p \vee q) - g(q)$. Table 2 shows the possible sign patterns of those values for a submodular function.

Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. We call g *quasisubmodular* if it satisfies (QSB):

(QSB) For all $p, q \in \mathbf{Z}^V$ we have $g(p \wedge q) \leqslant g(p)$ or $g(p \vee q) \leqslant g(q)$,

and call g *quasi L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (QSB) and (TRF). Since p and q are interchangeable, (QSB) implies $g(p \wedge q) \leqslant g(q)$ or $g(p \vee q) \leqslant g(p)$. Similarly, we call g *semistrictly quasisubmodular* if it satisfies (SSQSB):

(SSQSB) For all $p, q \in \mathbf{Z}^V$ we have both

(i) $g(p \vee q) \geqslant g(q) \Rightarrow g(p \wedge q) \leqslant g(p)$, and
(ii) $g(p \wedge q) \geqslant g(p) \Rightarrow g(p \vee q) \leqslant g(q)$,

and call g *semistrictly quasi L-convex* if $\mathrm{dom}\, g \neq \emptyset$ and it satisfies (SSQSB) and (TRF).

We also consider weaker properties than (QSB) and (SSQSB) by keeping in mind the possible sign patterns of the four values $g(p \wedge q) - g(p)$, $g(p \wedge q) - g(q)$, $g(p \vee q) - g(p)$, and $g(p \vee q) - g(q)$.

($\mathrm{QSB_w}$) For all $p, q \in \mathrm{dom}\, g$, we have $\max\{g(p), g(q)\} \geqslant \min\{g(p \wedge q), g(p \vee q)\}$.

($\mathrm{SSQSB_w}$) For all $p, q \in \mathrm{dom}\, g$, we have either of (i) and (ii):

(i) $\max\{g(p), g(q)\} > \min\{g(p \wedge q), g(p \vee q)\}$,
(ii) $g(p) = g(q) = g(p \wedge q) = g(p \vee q)$.

The property ($\mathrm{SSQSB_w}$) says that either (i) at least one of the values $g(p \wedge q) - g(p)$, $g(p \wedge q) - g(q)$, $g(p \vee q) - g(p)$, and $g(p \vee q) - g(q)$ is negative, or (ii) all the four values are equal to zero. Similarly, ($\mathrm{QSB_w}$) says that at least one of the four values is nonpositive.

The set version of quasisubmodularity can be obtained by translating the property (QSB) for the indicator function $\delta_D : \mathbf{Z}^V \to \{0, +\infty\}$ of a set $D \subseteq \mathbf{Z}^V$ in terms of D.

(QDL) $p, q \in D \Rightarrow p \wedge q \in D$ or $p \vee q \in D$.

The following properties for $D \subseteq \mathbf{Z}^V$ can be shown easily:

- (QDL) for $D \Leftrightarrow$ (QSB) for $\delta_D \Leftrightarrow$ ($\mathrm{QSB_w}$) for δ_D,
- (DL) for $D \Leftrightarrow$ (SSQSB) for $\delta_D \Leftrightarrow$ ($\mathrm{SSQSB_w}$) for δ_D.

We show some examples of quasi L-convex/submodular functions below.

Example 5.1. Let $\varphi : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$. We define $g : \mathbf{Z}^2 \to \mathbf{R} \cup \{+\infty\}$ by $g(p_1, p_2) = \varphi(p_1 - p_2)$ $((p_1, p_2) \in \mathbf{Z}^2)$. Then, g satisfies (TRF) with $r = 0$. By Theorem 2.2, g satisfies (QSB) (or ($\mathrm{QSB_w}$)) if and only if φ is quasiconvex, and g satisfies (SSQSB) (or ($\mathrm{SSQSB_w}$)) if and only if φ is semistrictly quasiconvex.

Example 5.2. Let $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a submodular function, and $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ be a nondecreasing function. We define the function $\tilde{g} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ by

$$\mathrm{dom}\,\tilde{g} = \mathrm{dom}\,g, \qquad \tilde{g}(p) = \varphi(g(p)) \quad (p \in \mathrm{dom}\,\tilde{g}). \tag{5.1}$$

Then, $\tilde{g}$ satisfies (QSB). Furthermore, if φ is strictly increasing, then $\tilde{g}$ satisfies (SSQSB). Note that if g satisfies (TRF) with $r = 0$, then $\tilde{g}$ also does.

Example 5.3. Let $D \subseteq \mathbf{Z}^V$ satisfy (DL), and $x \in \mathbf{R}^V$, $\alpha \in \mathbf{R}$. Then, the set $S = \{p \in D \mid \langle p, x \rangle \leqslant \alpha\}$ satisfies (QDL). Moreover, the function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\,g = S$ defined by $g(p) = \langle p, x \rangle$ $(p \in S)$ satisfies (SSQSB). These properties are obvious from the equation $\langle p, x \rangle + \langle q, x \rangle = \langle p \wedge q, x \rangle + \langle p \vee q, x \rangle$.

Remark 5.4. The concept of (semistrict) quasisubmodularity/L-convexity can be naturally extended to functions $g : S \to T$ with $S \subseteq \mathbf{Z}^V$ and a totally ordered set T with total order $\preccurlyeq$, as in the case of quasi M-convexity (see Remark 3.5). It is easy to see that the properties of (semistrictly) quasi L-convex functions shown in Sections 5 and 6.1 still hold true. For simplicity and convenience, however, we assume in this paper that the codomain of a function is $\mathbf{R} \cup \{+\infty\}$.

Example 5.5. Suppose that $V = \{1, 2, \ldots, n\}$ and put $V' = \{1, \ldots, n-1\}$. Let $a : V' \to \mathbf{Z} \cup \{-\infty\}$, $b : V' \to \mathbf{Z} \cup \{+\infty\}$ satisfy $a(i) \leqslant b(i)$ $(i \in V')$. For $i \in V'$, let $f_i : [a(i), b(i)] \to \mathbf{R}$ be a semistrictly quasiconvex function. We put $D = \{p \in \mathbf{Z}^V \mid a(i) \leqslant p(i) - p(n) \leqslant b(i)\ (i \in V')\}$ and define $g : D \to \mathbf{R}^{V'}$ by $g(p) = (g_i(p(i) - p(n)) \mid i \in V')$ $(p \in D)$, where the total order $\preccurlyeq$ on the codomain $\mathbf{R}^{V'}$ of g is given by the lexicographic order. Then, g satisfies (TRF) with $r = 0$ and (SSQSB) in the extended sense (see Remark 5.4). This fact can be shown similarly to that for Example 3.6.

The relationship among various versions of quasisubmodularity is summarized as follows.

Theorem 5.6. *For a function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, *we have*

$$\begin{array}{ccccc} \text{(SBM)} & \Rightarrow & \text{(SSQSB)} & \Rightarrow & \text{(QSB)} \\ & & \Downarrow & & \Downarrow \\ & & (\text{SSQSB}_\text{w}) & \Rightarrow & (\text{QSB}_\text{w}). \end{array}$$

Due to the definitions of quasi L-convexity/submodularity, most of the properties of quasisubmodular functions can be naturally restated in terms of quasi L-convex functions, and vice versa. In the following sections, we state properties mainly in terms of quasisubmodular functions and omit those for quasi L-convex functions whenever the restatements are immediate.

5.2. Level sets of quasi L-convex and submodular functions

We show that level sets of quasisubmodular functions have nice properties such as (DL) and (QDL). Furthermore, the weaker version of quasisubmodularity (QSB_w) for functions can be characterized by the property (QDL) of level sets.

Theorem 5.7. *A function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (QSB_w) *if and only if the level set* $\mathrm{L}(g, \alpha)$ *satisfies* (QDL) *for every* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *In particular, if* g *satisfies* (QSB_w), *then* $\mathrm{dom}\, g$ *and* $\arg\min g$ *satisfy* (QDL).

Proof. We show the "if" part only. Let $p, q \in \mathrm{dom}\, g$, and put $\alpha = \max\{g(p), g(q)\}$. Since $p, q \in \mathrm{L}(g, \alpha)$, we have $p \wedge q \in \mathrm{L}(g, \alpha)$ or $p \vee q \in \mathrm{L}(g, \alpha)$, i.e., $\max\{g(p), g(q)\} \geqslant \min\{g(p \wedge q), g(p \vee q)\}$. $\square$

Theorem 5.8. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$.

(i) *If the level set* $\mathrm{L}(g, \alpha)$ *satisfies* (DL) *for every* $\alpha \in \mathbf{R} \cup \{+\infty\}$, *then* g *satisfies* (QSB).
(ii) *If* g *satisfies* (SSQSB_w), *then* $\arg\min g$ *satisfies* (DL).

A submodular function over the integer lattice can be characterized by using level sets of functions perturbed by linear functions. Recall the definition of $g[x] : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ in (3.7).

Theorem 5.9 (Milgrom and Shannon [7, Theorem 10]). *A function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfies* (SBM) *if and only if for all* $x \in \mathbf{R}^V$ *and* $\alpha \in \mathbf{R}$ *the level set* $\mathrm{L}(g[x], \alpha)$ *satisfies* (QDL).

Proof. The "only if" part follows from Theorem 5.7 and submodularity of $g[x]$. We prove the "if" part. Let $p, q \in \mathrm{dom}\, g$. From (QDL) for $\mathrm{L}(g, \max\{g(p), g(q)\})$, we may assume $p \wedge q \in \mathrm{dom}\, g$ and $p \wedge q \neq p, q$. For any $\varepsilon > 0$, we can choose some $x \in \mathbf{R}^V$ and $\alpha \in \mathbf{R}$ with $\alpha = g[x](p) = g[x](q) = g[x](p \wedge q) - \varepsilon$. By (QDL) for $\mathrm{L}(g[x], \alpha)$, we have $p \vee q \in \mathrm{L}(g[x], \alpha)$, implying

$$g[x](p) + g[x](q) = 2\alpha \geqslant g[x](p \wedge q) + g[x](p \vee q) - \varepsilon.$$

Since ε can be chosen arbitrarily, we have $g[x](p)+g[x](q) \geqslant g[x](p\wedge q)+g[x](p\vee q)$, which is equivalent to the submodular inequality for p and q. $\square$

Combining Theorems 5.7 and 5.9, we see the following:

Corollary 5.10. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$. *Then,* g *satisfies* (SBM) $\Leftrightarrow \forall x \in \mathbf{R}^V$, $g[x]$ *satisfies* (QSB) $\Leftrightarrow \forall x \in \mathbf{R}^V$, $g[x]$ *satisfies* ($\mathrm{QSB_w}$).

5.3. Operations for quasi L-convex and submodular functions

The class of (semistrictly) quasi L-convex/submodular functions is closed under several fundamental operations.

Theorem 5.11. *Let* ($*\mathrm{QSB}_*$) *be one of the properties* (QSB), ($\mathrm{QSB_w}$), (SSQSB), and ($\mathrm{SSQSB_w}$), and $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *be a function with the property* ($*\mathrm{QSB}_*$).

(i) *For any* $a \in \mathbf{Z}^V$, $\beta \in \mathbf{Z}$, *and* $v > 0$, *the function* $vg(a + \beta p)$ *satisfies* ($*\mathrm{QSB}_*$) *as a function in* p.
(ii) *For any* $U \subseteq V$, *the function* $g^U : \mathbf{Z}^U \to \mathbf{R} \cup \{\pm\infty\}$ *defined by* $g^U(p) = \inf\{g(p, q) \mid q \in \mathbf{Z}^{V \setminus U}\}$ $(p \in \mathbf{Z}^U)$ *satisfies* ($*\mathrm{QSB}_*$) *if* $g^U > -\infty$.
(iii) *For any* $a : V \to \mathbf{Z} \cup \{-\infty\}$ *and* $b : V \to \mathbf{Z} \cup \{+\infty\}$ *with* $a \leqslant b$, *the function* $g_a^b : \mathbf{Z}^V \to \mathbf{Z} \cup \{+\infty\}$ *defined by* (3.8) *satisfies* ($*\mathrm{QSB}_*$).
(iv) *For* $i = 1, 2$, *let* V_i *be a finite set, and assume* ($*\mathrm{QSB}_*$) *for* $g_i : \mathbf{Z}^{V_i} \to \mathbf{R}_{++} \cup \{+\infty\}$. *Then, the function* $g : \mathbf{Z}^{V_1} \times \mathbf{Z}^{V_2} \to \mathbf{R}_{++} \cup \{+\infty\}$ *defined by* $g(p_1, p_2) = g_1(p_1) g_2(p_2)$ $(p_i \in \mathbf{Z}^{V_i},\ i = 1, 2)$ *satisfies* ($*\mathrm{QSB}_*$).

Remark 5.12. The class of (semistrictly) quasisubmodular functions is not closed under addition; it is not closed under the addition of a linear function.

Theorem 5.13. *For* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, *define* $\tilde{g} : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *by* (5.1).

(i) *If* g *satisfies* (QSB) (*resp.* ($\mathrm{QSB_w}$)) *and* φ *is nondecreasing, then* $\tilde{g}$ *also satisfies* (QSB) (*resp.* ($\mathrm{QSB_w}$)).
(ii) *If* g *satisfies* (SSQSB) (*resp.* ($\mathrm{SSQSB_w}$)) *and* φ *is strictly increasing, then* $\tilde{g}$ *also satisfies* (SSQSB) (*resp.* ($\mathrm{SSQSB_w}$)).

Theorem 5.14. *Let* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *and* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{-\infty\}$ *be functions with* $g(p) > 0$ $(\forall p \in \mathrm{dom}\, f)$. *Suppose that the function* $f(\cdot) - \alpha g(\cdot)$ *satisfies* ($\mathrm{QSB_w}$) *for all* $\alpha \in \mathbf{R} \cup \{+\infty\}$. *Then, the function* $r : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, r = \mathrm{dom}\, f$ *given* *by* $r(p) = f(p)/g(p)$ $(p \in \mathrm{dom}\, r)$ *also satisfies* (QSB$_\mathrm{w}$). *In particular, if* f *and* $-g$ *satisfy* (SBM), *then* r *satisfies* (QSB$_\mathrm{w}$).

Proof. The proof is clear from Theorem 5.7. □

6. Minimization of quasi L-convex functions

In this section, we consider the minimization of quasi L-convex functions. Throughout this section we assume $r = 0$ in (TRF) since otherwise quasi L-convex functions have no minimizer. Under this assumption, the minimization of a function $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ is equivalent to the minimization of $g_0 : \mathbf{Z}^{V\setminus\{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ which is defined as

$$g_0(p') = g(0, p') \quad ((0, p') \in \mathbf{Z} \times \mathbf{Z}^{V\setminus\{v_0\}}) \tag{6.1}$$

with an element $v_0 \in V$.

6.1. Properties of minimizers of quasi L-convex functions

Global minimality of quasi L-convex functions is characterized by local minimality.

Lemma 6.1. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfy* (TRF) *with* $r = 0$.

(i) *Assume* (QSB$_\mathrm{w}$) *for* g. *Then, for all* $p, q \in \mathbf{Z}^V$ *and* $\lambda \in \mathbf{Z}$ *we have*

$$\max\{g(p), g(q)\} \geqslant \min\{g(p \vee (q - \lambda\mathbf{1})), g((p + \lambda\mathbf{1}) \wedge q)\}. \tag{6.2}$$

In particular, for all $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$ *we have*

$$\max\{g(p), g(q)\} \geqslant \min\{g(p + \lambda\chi_X), g(q - \lambda\chi_X)\}, \tag{6.3}$$

where $X \subseteq V$, $\lambda_1 \in \mathbf{Z}$, *and* $\lambda_2 \in \mathbf{Z} \cup \{-\infty\}$ *are defined by*

$$X = \arg\max_{v \in V}\{q(v) - p(v)\}, \tag{6.4}$$

$$\lambda_1 = \max_{v \in V}\{q(v) - p(v)\}, \quad \lambda_2 = \max_{v \in V\setminus X}\{q(v) - p(v)\}. \tag{6.5}$$

(ii) *Assume* (SSQSB$_\mathrm{w}$) *for* g. *Then, for all* $p, q \in \mathbf{Z}^V$ *with* $g(p) \neq g(q)$ *and* $\lambda \in \mathbf{Z}$ *we have inequality* (6.2) *with strict inequality. In particular, for all* $p, q \in \mathrm{dom}\, g$ *with* $g(p) \neq g(q)$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$ *we have* (6.3) *with strict inequality.*

(iii) *Assume* (SSQSB) *for* g. *Then, for all* $p, q \in \mathbf{Z}^V$ *and* $\lambda \in \mathbf{Z}$ *we have the following properties*:

$$g(p \vee (q - \lambda\mathbf{1})) \geqslant g(p) \;\Rightarrow\; g((p + \lambda\mathbf{1}) \wedge q) \leqslant g(q),$$

$$g((p + \lambda\mathbf{1}) \wedge q) \geqslant g(q) \;\Rightarrow\; g(p \vee (q - \lambda\mathbf{1})) \leqslant g(p).$$

In particular, for all $p, q \in \mathrm{dom}\, g$ *and* $\lambda \in [0, \lambda_1 - \lambda_2]$ *we have*

$$g(p + \lambda\chi_X) \geqslant g(p) \quad\Rightarrow\quad g(q - \lambda\chi_X) \leqslant g(q),$$
$$g(q - \lambda\chi_X) \geqslant g(q) \quad\Rightarrow\quad g(p + \lambda\chi_X) \leqslant g(p), \tag{6.6}$$

where $X \subseteq V$, $\lambda_1 \in \mathbf{Z}$, *and* $\lambda_2 \in \mathbf{Z} \cup \{-\infty\}$ *are given by* (6.4) *and* (6.5).

Proof. Inequality (6.2) can be shown as follows:

$$\begin{aligned}\text{LHS of}(6.2) &= \max\{g(p), g(q - \lambda\mathbf{1})\}\\ &\geqslant \min\{g(p \vee (q - \lambda\mathbf{1})), g(p \wedge (q - \lambda\mathbf{1}))\}\\ &= \min\{g(p \vee (q - \lambda\mathbf{1})), g(p \wedge (q - \lambda\mathbf{1}) + \lambda\mathbf{1})\}\\ &= \text{RHS of } (6.2).\end{aligned}$$

Inequality (6.3) is obvious from (6.2) since $p \vee \{q - (\lambda_1 - \lambda)\mathbf{1}\} = p + \lambda\chi_X$ and $(p + (\lambda_1 - \lambda)\mathbf{1}) \wedge q = q - \lambda\chi_X$ for $\lambda \in [0, \lambda_1 - \lambda_2]$. Proofs of (ii) and (iii) are similar to that for (i). □

Theorem 6.2. *Let* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfy* (TRF) *with* $r = 0$, *and* $p \in \mathrm{dom}\, g$.

(i) *Assume* ($\mathrm{QSB_w}$) *for* g. *Then,* $g(p) < g(q)$ *for all* $q \in \mathbf{Z}^V$ *such that* $q - p$ *is not a multiple of* $\mathbf{1}$ *if and only if* $g(p) < g(p + \chi_X)$ *for all* $X \subseteq V$ *with* $X \notin \{\emptyset, V\}$.
(ii) *Assume* ($\mathrm{SSQSB_w}$) *for* g. *Then,*

$$g(p) \leqslant g(q) \ (\forall q \in \mathbf{Z}^V) \quad\Leftrightarrow\quad g(p) \leqslant g(p + \chi_X) \ (\forall X \subseteq V).$$

Proof. We prove the "if" part of (i) by contradiction. Suppose that $g(q) \leqslant g(p)$ holds for some $q \in \mathrm{dom}\, g$ such that $q - p$ is not a multiple of $\mathbf{1}$. We may assume $q \geqslant p$ by (TRF) for g, and also assume that q minimizes the value $\max_{v\in V} \{q(v) - p(v)\}$ among all such vectors. Put $X = \arg\max_{v\in V}\{q(v) - p(v)\}$, where $X \neq V$. By applying Lemma 6.1 to p and q, we obtain $g(p) = \max\{g(p), g(q)\} \geqslant \min\{g(p + \chi_X), g(q - \chi_X)\}$. Due to the choice of q, we have $g(p) < g(q - \chi_X)$. Hence, $g(p) \geqslant g(p + \chi_X)$ follows, a contradiction to the strict local minimality of p. The "only if" part of (i) is obvious, and (ii) can be shown similarly by Lemma 6.1(ii). □

Corollary 6.3. *For a function* $g : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *satisfying* (TRF) *with* $r = 0$, *define* $g_0 : \mathbf{Z}^{V\setminus\{v_0\}} \to \mathbf{R} \cup \{+\infty\}$ *by* (6.1). *Let* $p \in \mathrm{dom}\, g_0$.

(i) *Assume* ($\mathrm{QSB_w}$) *for* g. *Then,*

$$g_0(p) < g_0(q) \ (\forall q \in \mathbf{Z}^{V\setminus\{v_0\}} \setminus \{p\}) \Leftrightarrow g_0(p) < g_0(p \pm \chi_X) \, (\emptyset \neq \forall X \subseteq V\setminus\{v_0\}).$$

(ii) *Assume* ($\mathrm{SSQSB_w}$) *for* g. *Then,*

$$g_0(p) \leqslant g_0(q) \ (\forall q \in \mathbf{Z}^{V\setminus\{v_0\}}) \Leftrightarrow g_0(p) \leqslant g_0(p \pm \chi_X) \ (\forall X \subseteq V\setminus\{v_0\}).$$

We apply the scaling technique to the minimization of quasi L-convex functions in Section 6.2. Let $g: \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a semistrictly quasi L-convex function and α be any positive integer. Let $p_\alpha \in \operatorname{dom} g$ be an approximate minimum of g in the sense that p_α satisfies

$$g(p_\alpha) \leqslant g(p_\alpha + \alpha\chi_X) \quad (\forall X \subseteq V). \tag{6.7}$$

The following is a proximity theorem showing that a global minimum of a semistrictly L-convex function exists in the neighborhood of p_α. This generalizes an observation in [6].

Theorem 6.4. *Let $g: \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ be a function satisfying* (SSQSB) *and* (TRF) *with $r = 0$, and $\alpha \in \mathbf{Z}_{++}$. Suppose that $p_\alpha \in \operatorname{dom} g$ satisfies* (6.7). *Then,* $\arg\min g \neq \emptyset$ *and there exists some $q_* \in \arg\min g$ with*

$$p_\alpha \leqslant q_* \leqslant p_\alpha + (n-1)(\alpha - 1)\mathbf{1}. \tag{6.8}$$

Proof. It suffices to show that for any $\gamma \in \mathbf{R}$ with $\gamma > \inf g$, there exists some $q_* \in \operatorname{dom} g$ satisfying $g(q_*) \leqslant \gamma$ and (6.8). Assume, w.l.o.g., $p_\alpha = \mathbf{0}$. By (TRF) for g, there exists some $q_* \in \operatorname{dom} g$ such that $g(q_*) \leqslant \gamma$ and $q_* \geqslant \mathbf{0}$. We assume that q_* is minimal (w.r.t. the partial order $\geqslant$) among all such vectors. This assumption implies $q_*(v) = 0$ for some $v \in V$, i.e., $\operatorname{supp}^+(q_*) \neq V$, and

$$g(q_* - \chi_X) > g(q_*) \quad (\forall X \subseteq \operatorname{supp}^+(q_*)). \tag{6.9}$$

Then, there exist some $X_i \subseteq \operatorname{supp}^+(q_*)$ $(i = 1, \ldots, k)$ and $\{\mu_i\}_{i=1}^k \subseteq \mathbf{Z}_{++}$ such that $\emptyset \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset V$ and $q_* = \sum_{i=1}^k \mu_i \chi_{X_i}$, where $k \in [0, n-1]$.

Claim 1. *For any $j = 1, \ldots, k$ and $\mu \in [0, \mu_j - 1]$, we have*

$$g\left(\sum_{i=1}^{j-1} \mu_i \chi_{X_i} + \mu\chi_{X_j}\right) > g\left(\sum_{i=1}^{j-1} \mu_i \chi_{X_i} + (\mu+1)\chi_{X_j}\right).$$

Proof. Put $p = \sum_{i=1}^{j-1} \mu_i \chi_{X_i} + \mu\chi_{X_j}$ and suppose $p \in \operatorname{dom} g$. Then, $\arg\max_{v \in V} \{q_*(v) - p(v)\} = X_j$. Since $X_j \subseteq \operatorname{supp}^+(q_*)$, we have $g(q_* - \chi_{X_j}) > g(q_*)$ by (6.9). This fact, together with (6.6), yields $g(p + \chi_{X_j}) < g(p)$. $\square$

Claim 2. $g(\mu\chi_{X_j}) > g((\mu+1)\chi_{X_j})$ $(j = 1, \ldots, k,\ \mu \in [0, \mu_j - 1])$.

Proof. Assume $\mu\chi_{X_j} \in \operatorname{dom} g$. Put $p = \sum_{i=1}^{j} \mu_i \chi_{X_i}$ and $q = \mu\chi_{X_j}$. Then, $\arg\max_{v \in V} \{q(v) - p(v)\} = V \setminus X_j$. Since $g(p + \chi_{V \setminus X_j}) = g(p - \chi_{X_j}) > g(p)$ by Claim 1, (6.6) implies $g(q) > g(q - \chi_{V \setminus X_j}) = g(q + \chi_{X_j})$. $\square$

From Claim 2 and (6.7) follows $\mu_i < \alpha$ for $i = 1, 2, \ldots, k$. Hence, we have $\mathbf{0} \leqslant q_* \leqslant (\alpha - 1)\sum_{i=1}^k \chi_{X_i} \leqslant (n-1)(\alpha-1)\mathbf{1}$.

Corollary 6.5. *Given a function* $g:\mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$ *satisfying* (SSQSB) *and* (TRF) *with* $r=0$, *define* $g_0:\mathbf{Z}^{V\setminus\{v_0\}} \to \mathbf{R}\cup\{+\infty\}$ *by* (6.1). *Let* $\alpha\in\mathbf{Z}_{++}$. *If* $p_\alpha\in\mathrm{dom}\,g_0$ *satisfies* $g_0(p_\alpha)\leqslant g_0(p_\alpha\pm\alpha\chi_X)$ $(\forall X\subseteq V\setminus\{v_0\})$, *then there exists some* $q_*\in\arg\min g_0$ *such that* $|q_*(v)-p_\alpha(v)|\leqslant(n-1)(\alpha-1)$ $(v\in V\setminus\{v_0\})$.

6.2. Algorithms

Let $g:\mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$ satisfy ($\mathrm{SSQSB_w}$) and (TRF) with $r=0$, and define $g_0:\mathbf{Z}^{V\setminus\{v_0\}} \to \mathbf{R}\cup\{+\infty\}$ by (6.1). We assume that an initial vector $p_0\in\mathrm{dom}\,g_0$ is given a priori, and that we have an oracle for computing a function value of g_0 in unit time.

Remark 6.6. It is difficult to find a vector in $\mathrm{dom}\,g_0$ efficiently even if $\mathrm{dom}\,g_0$ is a bounded set. See Remark 4.5.

By Corollary 6.3, we can find a minimizer of g_0 by a descent method.

Algorithm DESCENT_L
Step 0: Let p be any vector in $\mathrm{dom}\,g_0$.
Step 1: If $g_0(p)=\min\{g_0(p\pm\chi_X)\mid X\subseteq V\setminus\{v_0\}\}$ then stop. [p is a minimizer]
Step 2: Find $X\subseteq V\setminus\{v_0\}$ and $\lambda\in\{1,-1\}$ such that $g_0(p+\lambda\chi_X)<g_0(p)$.
Step 3: Set $p:=p+\lambda\chi_X$. Go to Step 1.

If $\mathrm{dom}\,g_0$ is bounded, DESCENT_L terminates in at most $|\mathrm{dom}\,g_0|\leqslant K^{n-1}$ iterations, where $K=\max\{|p(v)-q(v)|\mid p,q\in\mathrm{dom}\,g_0,\ v\in V\setminus\{v_0\}\}$. We note that DESCENT_L may require checking all vectors in $\mathrm{dom}\,g_0$ in the worst case.

We further assume (SSQSB) for g. Based on Corollary 6.5, we apply the scaling technique to DESCENT_L to obtain a faster algorithm.

Algorithm SCALING_DESCENT_L
Step 0: Put $\alpha:=2^{\lceil\log_2 K\rceil}$, $D:=\mathrm{dom}\,g_0$. Let p_* be any vector in $\mathrm{dom}\,g_0$.
Step 1: Find $q\in\mathbf{Z}^{V\setminus\{v_0\}}$ such that $p_*+\alpha q\in D$ and
$g_0(p_*+\alpha q)=\min\{g_0(p_*+\alpha q')\mid q'\in\mathbf{Z}^{V\setminus\{v_0\}},\ p_*+\alpha q'\in D\}$.
Step 2: If $\alpha=1$ then stop. [$p_*+\alpha q$ is a minimizer of g_0.]
Step 3: Put $p_*:=p_*+\alpha q$,
$D:=D\cap\{p\in V\setminus\{v_0\}\mid |p(v)-p_*(v)|\leqslant(n-1)(\alpha-1)\ (v\in V\setminus\{v_0\})\}$,
and $\alpha:=\alpha/2$. Go to Step 1.

The number of scaling phases is $\lceil\log_2 K\rceil$. Therefore, if we could perform Step 1 in each iteration in polynomial time, SCALING_DESCENT_L would run in polynomial time. Unfortunately, we do not know yet such a polynomial-time algorithm for Step 1.

7. Concluding remarks

As observed in the previous sections, quasi M-/L-convexity inherit nice properties from M-/L-convexity. We here show by examples that some other properties of M-/L-convex functions do not extend to quasi M-/L-convex functions.

Failure of separation theorems: A (discrete) separation theorem holds for a pair of M-convex/M-concave functions and a pair of L-convex/L-concave functions.

Theorem 7.1 (Murota [10,11]). *Let* $f:\mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$, $g:\mathbf{Z}^V \to \mathbf{R}\cup\{-\infty\}$ *be functions with* $\mathrm{dom}\, f \cap \{x\in\mathbf{Z}^V \mid g(x) > -\infty\} \neq \emptyset$ *and* $f(x) \geqslant g(x)$ $(x\in\mathbf{Z}^V)$.

(i) *If* f *and* $-g$ *are M-convex functions, then there exist* $\alpha_* \in \mathbf{R}$ *and* $p_* \in \mathbf{R}^V$ *such that* $f(x) \geqslant \alpha_* + \langle p_*, x\rangle \geqslant g(x)$ $(x\in\mathbf{Z}^V)$.
(ii) *If* f *and* $-g$ *are L-convex functions, then there exist* $\alpha_* \in \mathbf{R}$ *and* $p_* \in \mathbf{R}^V$ *such that* $f(x) \geqslant \alpha_* + \langle p_*, x\rangle \geqslant g(x)$ $(x\in\mathbf{Z}^V)$.

The following example, on the other hand, shows that the separation theorem does not hold for a pair of quasi M-convex/M-concave functions.

Define $f:\mathbf{Z}^2 \to \mathbf{Z}\cup\{+\infty\}$ and $g:\mathbf{Z}^2 \to \mathbf{Z}\cup\{-\infty\}$ by

$$\mathrm{dom}\, f = \{(0,0),(1,-1),(2,-2)\},$$

$$f(0,0)=0, \quad f(1,-1)=2, \quad f(2,-2)=3,$$

$$\mathrm{dom}\, g = \{(1,-1),(2,-2)\}, \quad g(1,-1)=2, \quad g(2,-2)=3. \tag{7.1}$$

It is easy to check that f and $-g$ satisfy (SSQM) and that there is no $\alpha_* \in \mathbf{R}$ and $p_* \in \mathbf{R}^2$ satisfying $f(x) \geqslant \alpha_* + \langle p_*, x\rangle \geqslant g(x)$ $(x\in\mathbf{Z}^2)$. We can construct a similar example of a pair of quasi L-convex/L-concave functions for which the separation theorem does not hold.

Emptyness of subdifferentials: For a function $f:\mathbf{Z}^V \to \mathbf{R}\cup\{+\infty\}$ and $x\in\mathrm{dom}\, f$, the *subdifferential* of f at x, denoted by $\partial f(x)$, is defined by $\partial f(x)=\{p\in\mathbf{R}^V \mid f(y)-f(x) \geqslant \langle p, y-x\rangle\ (\forall y\in\mathbf{Z}^V)\}$. If f is an M-convex function, then $\partial f(x) \neq \emptyset$ for any $x\in\mathrm{dom}\, f$ [10,11]. For quasi M-convex functions, however, the subdifferential $\partial f(x)$ can be empty. For example, let $f:\mathbf{Z}^2 \to \mathbf{R}\cup\{+\infty\}$ be the function given by (7.1). Although f satisfies (SSQM), $\partial f(x)$ is empty for $x=(1,-1)$. We can construct a similar example of a quasi L-convex function for which the subdifferential is empty.

Conjugacy relationship: Given an integer-valued function $f:\mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$, we define the *conjugate* $f^\bullet:\mathbf{Z}^V \to \mathbf{Z}\cup\{\pm\infty\}$ of f by $f^\bullet(p)=\sup_{x\in\mathbf{Z}^V}\{\langle p,x\rangle - f(x)\}$ $(p\in\mathbf{Z}^V)$. It is shown in [10,11] that the conjugate of an integer-valued M-convex function is an integer-valued L-convex function, and vice versa. This nice conjugacy relationship between M- and L-convexity, however, does not extend to quasi M-/L-convexity.

For example, let $f:\mathbf{Z}^3 \to \mathbf{Z}\cup\{+\infty\}$ be a function defined by

$$\mathrm{dom}\, f = \{x\in\mathbf{Z}^3 \mid x_1+x_2+x_3=0\},$$

$$f(x_1,x_2,x_3) = \max\{x_1, x_2, x_1+x_2-1, -1\} \quad (x\in\mathrm{dom}\, f).$$

We can show that f satisfies (QM), i.e., f is quasi M-convex. The conjugate $g = f^{\bullet}$ is given by

$$g(p_1, p_2, p_3) = \begin{cases} +1 & (p \in \mathbf{Z}^3,\ (p_1 - p_3, p_2 - p_3) \in \{(0,0),(1,1)\}), \\ 0 & (p \in \mathbf{Z}^3,\ (p_1 - p_3, p_2 - p_3) \in \{(1,0),(0,1)\}), \\ +\infty & (\text{otherwise}), \end{cases}$$

which does not satisfy ($\mathrm{QSB_w}$), i.e., g is not quasi L-convex.

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